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RESEARCH ARTICLE

On Pointwise Lacunary Statistical Convergence of Order α of Sequences of Function

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Abstract In this paper we introduce the concepts of pointwise lacunary statistical convergence of order α and pointwise $w_p(f,\theta)$—summability of order α of sequences of real valued functions. Also some relations between pointwise $S_\theta^\alpha(f)$—statistical convergence and pointwise $w_p^\alpha(f,\theta)$—summability are given.

Keywords Statistical convergence · Sequences of function · Cesàro summability

Mathematics Subject Classification 40A05 · 40C05 · 46A45

1 Introduction

The concept of statistical convergence was introduced by Steinhaus [1] and Fast [2] and later reintroduced by Schoenberg [3] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Caserta et al. [4], Çakallı [5, 6], Çakallı and Khan [7], Connor [8], Et et al. [9, 10], Fridy [11], Güngör et al. [12], Kolk [13], Mursaleen [14], Salat [15], Tripathy et al. [16–18] and many others.

The definition of pointwise statistical convergence of sequences of real valued functions was given by Gökhan and Güngör [19] and independently by Duman and Orhan [20].

In this paper we introduce and examine the concepts of pointwise lacunary statistical convergence of order α and pointwise $w_p(f,\theta)$—summability of order α of sequences of real valued functions.

2 Definition and Preliminaries

The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The asymptotic density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k) \text{ provided the limit exists,}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c)=1-\delta(E)$.

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan in [21] and after that statistical convergence of order α and strong p-Cesàro summability of order α was studied by Çolak [22].

By a lacunary sequence we mean an increasing integer sequence $\theta=(k_r)$ such that $h_r=(k_r-k_{r-1})\to\infty$ as $r\to\infty$. Throught this paper the intervals determined by θ is denoted by $I_r=(k_{r-1},k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is abbreviated by q_r. Recently lacunary sequence have been studied by several authors [23–33].

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Definition 2.1 Let $\theta = (k_r)$ be a lacunary sequence and $\alpha \in (0,1]$ be any real number. A sequence of functions $\{f_k\}$ is said to be pointwise $S_\theta^\alpha(f)$—statistically convergent (or pointwise lacunary statistical convergence of order α) to the function f on a set A, if for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r^\alpha} |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}| = 0$$

where $I_r = (k_{r-1}, k_r]$ and $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \ldots, h_r^\alpha, \ldots)$. In this case we write $S_\theta^\alpha - \lim f_k(x) = f(x)$ on A. $S_\theta^\alpha - \lim f_k(x) = f(x)$ means that for every $\delta > 0$ and $0 < \alpha \leq 1$, there is an integer n_0 such that

$$\frac{1}{h_r^\alpha} |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}| < \delta,$$

for all $n > n_0 (= n_0(\varepsilon, \delta, x))$ and for each $\varepsilon > 0$. The set of all pointwise lacunary statistical convergence of order α will be denoted by $S_\theta^\alpha(f)$. For $\theta = (2^r)$, we shall write $S^\alpha(f)$ instead of $S_\theta^\alpha(f)$ which were defined and studied by Çınar et al. [34] and in the special case $\alpha = 1$, we write $S_\theta(f)$ instead of $S_\theta^\alpha(f)$.

Pointwise lacunary statistical convergence of order α of sequence of functions is well defined for $0 < \alpha \leq 1$, but is not well defined for $\alpha > 1$. For this let $\{f_k\}$ be defined as follows:

$$f_k(x) = \begin{cases} 1 & k = 2r \\ \dfrac{kx}{1 + k^2x^2} & k \neq 2r \end{cases} \quad r = 1, 2, 3 \ldots, x \in \left[0, \frac{1}{2}\right]$$

Then, both

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |f_k(x) - 1| \geq \varepsilon, \quad \text{for every } x \in A\}| \leq \lim_{r\to\infty} \frac{k_r - k_{r-1}}{2h_r^\alpha} = \lim_{r\to\infty} \frac{h_r}{2h_r^\alpha} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |f_k(x) - 0| \geq \varepsilon, \quad \text{for every } x \in A\}| \leq \lim_{r\to\infty} \frac{k_r - k_{r-1}}{2h_r^\alpha} = \lim_{r\to\infty} \frac{h_r}{2h_r^\alpha} = 0$$

for $\alpha > 1$, and so $S_\theta^\alpha - \lim f_k(x) = 1$ and $S_\theta^\alpha - \lim f_k(x) = 0$.

It is easy to see that every convergent sequence of functions is statistically convergent of order α $(0 < \alpha \leq 1)$. The following example shows that the converse of this does not hold. The sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 1 & k = n^3 \\ \dfrac{2kx}{1 + k^2x^2} & k \neq n^3 \end{cases}$$

is statistically convergent of order α with $S^\alpha - \lim f_k(x) = 0$ for $\alpha > \frac{1}{3}$, but it is not convergent.

Definition 2.2 Let $\theta = (k_r)$ be a lacunary sequence, $\alpha \in (0,1]$ and p be a positive real number. A sequence of functions $\{f_k\}$ is said to be pointwise $w_p^\alpha(f, \theta)$—summable (or pointwise $w_p(f, \theta)$—summable of order α), if there is a function f such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r, x \in A} |f_k(x) - f(x)|^p = 0.$$

In this case we write $w_p^\alpha(f, \theta) - \lim f_k(x) = f(x)$ on A. The set of all pointwise $w_p(f, \theta)$—summable sequence of functions order α will be denoted by $w_p^\alpha(f, \theta)$.

Let A be any non empty set, by $B(A)$ we denote the set of all bounded real valued functions defined on A.

3 Main Results

Theorem 3.1 *Let $\theta = (k_r)$ be a lacunary sequence, $\alpha \in (0,1]$ be any real number and $\{f_k\}, \{g_k\}$ be sequences of real valued functions defined on a set A.*

(i) If $S_\theta^\alpha - \lim f_k(x) = f(x)$ and $c \in \mathbb{R}$, then $S_\theta^\alpha - \lim cf_k(x) = cf(x)$,

(ii) If $S_\theta^\alpha - \lim f_k(x) = f(x)$ and $S_\theta^\alpha - \lim g_k(x) = g(x)$, then $S_\theta^\alpha - \lim (f_k(x) + g_k(x)) = f(x) + g(x)$.

Theorem 3.2 *Let $\theta = (k_r)$ be a lacunary sequence and $\alpha, \beta \in (0,1]$ $(\alpha \leq \beta)$. Then $S_\theta^\alpha(f) \subseteq S_\theta^\beta(f)$ and the inclusion is strict for the case $\alpha < \beta$.*

Proof To show the strictness of the inclusion $S_\theta^\alpha(f) \subseteq S_\theta^\beta(f)$, let us define a sequence $\{f_k\}$ by

$$f_k(x) = \begin{cases} 1 & k = r^2 \\ \dfrac{kx + 2}{1 + k^2x^2} & k \neq r^2 \end{cases}, \quad x \in [2, 3].$$

Then $x \in S_\theta^\beta(f)$ for $\frac{1}{2} < \beta \leq 1$, but $x \notin S_\theta^\alpha(f)$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 3.3 *Let $\theta = (k_r)$ be a lacunary sequence, $0 < \alpha \leq \beta \leq 1$ and p be a positive real number. Then $w_p^\alpha(f, \theta) \subseteq w_p^\beta(f, \theta)$ and the inclusion is strict for the case $\alpha < \beta$.*

Proof Taking $\theta = (2^r)$ we show the strictness of the inclusion $w_p^\alpha(f, \theta) \subseteq w_p^\beta(f, \theta)$ for a special case. For this consider the sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} \dfrac{k^2x^2}{1 + k^2x^2} & k = n^2 \\ 0 & k \neq n^2 \end{cases}, \quad x \in [1, 2].$$

Then

$$\frac{1}{n^\beta} \sum_{k=1}^{n} |f_k(x) - 0|^p \leq \frac{\sqrt{n}}{n^\beta} = \frac{1}{n^{\beta - \frac{1}{2}}} \to 0 \text{ as n} \to \infty$$

and

$$\frac{1}{n^\alpha}\sum_{k=1}^{n}|f_k(x)-0|^p \geq \frac{\sqrt{n}}{2n^\alpha} \to \infty \text{ as n} \to \infty$$

and so the sequence $\{f_k\}$ is pointwise $w_p(f,\theta)$—summable of order β for $\frac{1}{2}<\beta\leq 1$, but is not pointwise $w_p(f,\theta)$—summable of order α for $0<\alpha<\frac{1}{2}$. □

The following result is established using standard techniques, so we state the result without proof.

Theorem 3.4 *Let $\theta=(k_r)$ be a lacunary sequence and let α and β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$ and $0<p<\infty$. If a sequence of functions $\{f_k\}$ is pointwise $w_p(f,\theta)$—summable of order α, to the function f, then it is pointwise lacunary statistical convergence of order β, to the function f.*

Theorem 3.5 *Let $\theta=(k_r)$ be a lacunary sequence and $\alpha\in(0,1]$. If $\lim\inf_r q_r>1$ then $S^\alpha(f)\subset S^\alpha_\theta(f)$.*

Proof Suppose that $\lim\inf_r q_r>1$, then there exists a $\delta>0$ such that $q_r\geq 1+\delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta}\Longrightarrow\left(\frac{h_r}{k_r}\right)^\alpha\geq\left(\frac{\delta}{1+\delta}\right)^\alpha\Longrightarrow\frac{1}{k_r^\alpha}\geq\frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}.$$

If $S^\alpha-\lim f_k(x)=f(x)$ on A, then for every $\varepsilon>0$ and for sufficiently large r, we have

$$\begin{aligned}&\frac{1}{k_r^\alpha}|\{k\leq k_r:|f_k(x)-f(x)|\geq\varepsilon,\ \text{for every } x\in A\}|\\ &\geq\frac{1}{k_r^\alpha}|\{k\in I_r:|f_k(x)-f(x)|\geq\varepsilon,\ \text{for every } x\in A\}|\\ &\geq\frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}|\{k\in I_r:|f_k(x)-f(x)|\geq\varepsilon,\ \text{for every } x\in A\}|;\end{aligned}$$

this proves the proof. □

Theorem 3.6 *Let $\theta=(k_r)$ be a lacunary sequence and $\alpha\in(0,1]$. If $\lim\sup_r q_r<\infty$ then $S^\alpha_\theta(f)\subset S(f)$.*

Proof If $\lim\sup_r q_r<\infty$, then there is an $H>0$ such that $q_r<H$ for all r. Suppose that $S^\alpha_\theta-\lim f_k(x)=f(x)$ on A and let $N_r=|\{k\in I_r:|f_k(x)-f(x)|\geq\varepsilon, \text{for every } x\in A\}|$. By the definition for a given $\varepsilon>0$, there is an $r_0\in\mathbb{N}$ such that for $0<\alpha\leq 1$,

$$\frac{N_r}{h_r^\alpha}<\varepsilon\Longrightarrow\frac{N_r}{h_r}<\varepsilon,\quad \text{for all r} > \text{r}_0.$$

The rest of proof follows from Lemma 3 in [26]. □

Theorem 3.7 *Let $\theta=(k_r)$ be a lacunary sequence and $\alpha\in(0,1]$. If $\lim\inf_r\frac{h_r^\alpha}{k_r}>0$ then $S(f)\subset S^\alpha_\theta(f)$.*

Proof For a given $\varepsilon>0$, we have

$$\begin{aligned}&\frac{1}{k_r}|\{k\leq k_r:|f_k(x)-f(x)|\geq\varepsilon,\quad\text{for every } x\in A\}|\\ &\geq\frac{1}{k_r}|\{k\in I_r:|f_k(x)-f(x)|\geq\varepsilon,\quad\text{for every } x\in A\}|\\ &=\frac{h_r^\alpha}{k_r}\frac{1}{h_r^\alpha}|\{k\in I_r:|f_k(x)-f(x)|\geq\varepsilon,\quad\text{for every } x\in A\}|.\end{aligned}$$

Taking limit as $r\to\infty$, we get $S^\alpha_\theta-\lim f_k(x)=f(x)$ on A. □

The proofs of the following theorems are obtained by using the standard techniques.

Theorem 3.8 *Let $0<\alpha\leq 1$ and $\theta=(k_r)$ be a lacunary sequence. If $\lim\inf_r q_r>1$ then $w^\alpha_p(f)\subset w^\alpha_p(f,\theta)$.*

Theorem 3.9 *Let $\theta=(k_r)$ be a lacunary sequence. If $\lim\sup_r q_r<\infty$ then $w_p(f,\theta)\subseteq w_p(f)$.*

Theorem 3.10 *Let $0<\alpha\leq 1$ and $0<p<q<\infty$ then $w^\alpha_q(f,\theta)\subseteq w^\alpha_p(f,\theta)$.*

Let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subseteq J_r$ for all $r\in\mathbb{N}$, α and β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$. Now we give some inclusion relations between the sets of $S^\alpha_\theta(f)$—statistically convergent sequences and pointwise $w^\alpha(f,\theta)$—summable sequences for different $\alpha's$ and $\theta's$ which also include Theorems 3.2, 3.3 and 3.4 as a special case.

Theorem 3.11 *Let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subset J_r$ for all $r\in\mathbb{N}$ and let α and β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$*

(i) *If*

$$\lim_{r\to\infty}\inf\frac{h_r^\alpha}{\ell_r^\beta}>0 \tag{1}$$

then $S^\beta_{\theta'}(f)\subseteq S^\alpha_\theta(f)$,

(ii) *If*

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^\beta}=1 \tag{2}$$

then $S^\alpha_\theta(f)\subseteq S^\beta_{\theta'}(f)$, where $I_r=(k_{r-1},k_r]$, $J_r=(s_{r-1},s_r]$, $h_r=k_r-k_{r-1}$ and $\ell_r=s_r-s_{r-1}$.

Proof (*i*) Easy, so omitted.

(*ii*) Let $(f_k(x))\in S^\alpha_\theta(f)$ and Eq. (2) be satisfied. Since $I_r\subset J_r$, for $\varepsilon>0$ we may write

$$\frac{1}{\ell_r^{\beta}}|\{k \in J_r : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}|$$
$$= \frac{1}{\ell_r^{\beta}}|\{s_{r-1} < k \leq k_{r-1} : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}|$$
$$+ \frac{1}{\ell_r^{\beta}}|\{k_r < k \leq s_r : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}|$$
$$+ \frac{1}{\ell_r^{\beta}}|\{k_{r-1} < k \leq k_r : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}|$$
$$\leq \frac{k_{r-1} - s_{r-1}}{\ell_r^{\beta}} + \frac{s_r - k_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}}|\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon, \text{ for every } x \in A\}|$$
$$= \frac{\ell_r - h_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}}|\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}|$$
$$\leq \frac{\ell_r - h_r^{\beta}}{h_r^{\beta}} + \frac{1}{h_r^{\beta}}|\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}|$$
$$\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) + \frac{1}{h_r^{\alpha}}|\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every } x \in A\}|$$

for all $r \in \mathbb{N}$. This implies that $S_\theta^\alpha(f) \subseteq S_{\theta'}^\beta(f)$. □

From Theorem 3.11 we have the following results.

Corollary 3.11.1 *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$.*
If Eq. (1) holds then

(i) $S_{\theta'}^\alpha(f) \subseteq S_\theta^\alpha(f)$ *for each $\alpha \in (0,1]$ and for all $x \in A$,*
(ii) $S_{\theta'}(f) \subseteq S_\theta^\alpha(f)$ *for each $\alpha \in (0,1]$ and for all $x \in A$,*
(iii) $S_{\theta'}(f) \subseteq S_\theta(f)$ *and for all $x \in A$.*

If Eq. (2) holds then

(i) $S_\theta^\alpha(f) \subseteq S_{\theta'}^\alpha(f)$ *for each $\alpha \in (0,1]$ and for all $x \in A$,*
(ii) $S_\theta^\alpha(f) \subseteq S_{\theta'}(f)$ *for each $\alpha \in (0,1]$ and for all $x \in A$,*
(iii) $S_\theta(f) \subseteq S_{\theta'}(f)$ *for all $x \in A$.*

Theorem 3.12 *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. Then we have*

(i) *If Eq. (1) holds then $w_p^\beta(f,\theta') \subset w_p^\alpha(f,\theta)$ for all $x \in A$,*
(ii) *Let Eq. (2) holds, $f(x) \in B(A)$ and $\{f_k\}$ be a sequence of bounded real valued functions defined on a set A then $w_p^\alpha(f,\theta) \subset w_p^\beta(f,\theta')$ for all $x \in A$.*

Proof Suppose that Eq. (2) holds and $\{f_k\}$ be a sequence of bounded real valued functions defined on a set A. Since $f(x) \in B(A)$ then there exists some $M > 0$ such that $|f_k(x) - f(x)| \leq M$ for all $k \in \mathbb{N}$ and for all $x \in A$. Now, we may write

$$\frac{1}{\ell_r^{\beta}}\sum_{k \in J_r}|f_k(x) - f(x)|^p$$
$$= \frac{1}{\ell_r^{\beta}}\sum_{k \in J_r - I_r, x \in A}|f_k(x) - f(x)|^p + \frac{1}{\ell_r^{\beta}}\sum_{k \in I_r, x \in A}|f_k(x) - f(x)|^p$$
$$\leq \left(\frac{\ell_r - h_r}{\ell_r^{\beta}}\right)M^p + \frac{1}{\ell_r^{\beta}}\sum_{k \in I_r, x \in A}|f_k(x) - f(x)|^p$$
$$\leq \left(\frac{\ell_r - h_r^{\beta}}{h_r^{\beta}}\right)M^p + \frac{1}{h_r^{\beta}}\sum_{k \in I_r, x \in A}|f_k(x) - f(x)|^p$$
$$\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right)M^p + \frac{1}{h_r^{\alpha}}\sum_{k \in I_r, x \in A}|f_k(x) - f(x)|^p$$

for every $r \in \mathbb{N}$. Therefore $w_p^\alpha(f,\theta) \subset w_p^\beta(f,\theta')$. □

From Theorem 3.12 we have the following results.

Corollary 3.12.1 *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.*
If Eq. (1) holds then

(i) $w_p^\alpha(f,\theta') \subseteq w_p^\alpha(f,\theta)$ *for each $\alpha \in (0,1]$ and for all $x \in A$,*
(ii) $w_p(f,\theta') \subseteq w_p^\alpha(f,\theta)$ *for each $\alpha \in (0,1]$ and for all $x \in A$,*
(iii) $w_p(f,\theta') \subseteq w_p(f,\theta)$ *for all $x \in A$,*

Let Eq. (2) holds, $f(x) \in B(A)$ and $\{f_k\}$ be a sequence of bounded real valued functions defined on a set A, then

(i) $w_p^\alpha(f,\theta) \subset w_p^\alpha(f,\theta')$ *for each $\alpha \in (0,1]$ and for all $x \in A$,*
(ii) $w_p^\alpha(f,\theta) \subset w_p(f,\theta')$ *for each $\alpha \in (0,1]$ and for all $x \in A$,*
(iii) $w_p(f,\theta) \subset w_p(f,\theta')$ *for all $x \in A$.*

Theorem 3.13 *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. Then*

(i) *Let Eq. (1) holds, if a sequence of real valued functions defined on a set A is pointwise $w_p^\beta(f,\theta')$—summable to f, then it is pointwise lacunary statistical convergence of order α to the function f on a set A,*
(ii) *Let Eq. (2) holds, $f(x) \in B(A)$ and $\{f_k\}$ be a sequence of bounded real valued functions defined on a set A, if a sequence is pointwise lacunary statistical convergence of order α to the function f then it is pointwise $w_p^\beta(f,\theta')$—summable to f.*

Proof (*i*) Let $w_p^\alpha(f,\theta) - \lim f_k(x) = f(x)$ on A and $\varepsilon > 0$, then we have

$$\frac{1}{\ell_r^\beta}\sum_{k\in J_r, x\in A}|x_k-L|^p \geq \frac{h_r^\alpha}{\ell_r^\beta}\frac{1}{h_r^\alpha}|\{k\in I_r : |f_k(x)-f(x)|\geq \varepsilon \text{ for every } x\in A\}|\varepsilon^p.$$

Since Eq. (1) holds, the sequence $\{f_k\}$ is a pointwise lacunary statistically convergent sequence of order α to the function f on a set A.

(ii) Suppose that the sequence $\{f_k\}$ is a pointwise lacunary statistically convergent sequence of order α to the function f on a set A. Since $f(x)\in B(A)$ and $\{f_k\}$ is a bounded sequence of real valued functions defined on a set A, there exists a $M>0$ such that $|f_k(x)-f(x)|\leq M$ for all k. Then for every $\varepsilon>0$ we may write

$$\begin{aligned}
&\frac{1}{\ell_r^\beta}\sum_{k\in J_r, x\in A}|f_k(x)-f(x)|^p\\
&=\frac{1}{\ell_r^\beta}\sum_{k\in J_r-I_r, x\in A}|f_k(x)-f(x)|^p+\frac{1}{\ell_r^\beta}\sum_{k\in I_r, x\in A}|f_k(x)-f(x)|^p\\
&\leq\left(\frac{\ell_r-h_r}{\ell_r^\beta}\right)M^p+\frac{1}{\ell_r^\beta}\sum_{k\in I_r, x\in A}|f_k(x)-f(x)|^p\\
&\leq\left(\frac{\ell_r-h_r^\beta}{\ell_r^\beta}\right)M^p+\frac{1}{\ell_r^\beta}\sum_{k\in I_r, x\in A}|f_k(x)-f(x)|^p\\
&\leq\left(\frac{\ell_r}{h_r^\beta}-1\right)M^p+\frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ |f_k(x)-f(x)|\geq\varepsilon, x\in A}}|f_k(x)-f(x)|^p\\
&\quad+\frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ |f_k(x)-f(x)|<\varepsilon, x\in A}}|f_k(x)-f(x)|^p\\
&\leq\left(\frac{\ell_r}{h_r^\beta}-1\right)M^p+\frac{M^p}{h_r^\beta}\left|\left\{\begin{matrix}k\in I_r : |f_k(x)-f(x)|\geq\varepsilon\\ \text{for every } x\in A\end{matrix}\right\}\right|+\frac{h_r}{h_r^\beta}\varepsilon^p\\
&\leq\left(\frac{\ell_r}{h_r^\beta}-1\right)M^p+\frac{M^p}{h_r^\alpha}\left|\left\{\begin{matrix}k\in I_r : |f_k(x)-f(x)|\geq\varepsilon\\ \text{for every } x\in A\end{matrix}\right\}\right|+\frac{\ell_r}{h_r^\beta}\varepsilon^p
\end{aligned}$$

for all $r\in\mathbb{N}$. Using Eq. (2) we obtain that $w_p^\beta(f,\theta')-\lim f_k(x)=f(x)$, whenever $S_\theta^\alpha(f)-\lim f_k(x)=f(x)$. □

From Theorem 3.13 we have the following result.

Corollary 3.13.1 *Let α be any fixed real number such that $0<\alpha\leq 1, 0<p<\infty$ and let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subset J_r$ for all $r\in\mathbb{N}$.*

If Eq. (1) holds then

(i) *If a sequence of real valued functions defined on a set A is pointwise $w_p^\alpha(f,\theta')$—summable to f, then it is pointwise lacunary statistically convergent sequence of order α to the function f on a set A,*
(ii) *If a sequence of real valued functions defined on a set A is pointwise $w_p(f,\theta')$—summable to f, then it is pointwise lacunary statistically convergent sequence of order α to the function f on a set A,*
(iii) *If a sequence of real valued functions defined on a set A is pointwise $w_p(f,\theta')$—summable to f, then it is pointwise lacunary statistically convergent sequence to the function f on a set A,*

Let Eq. (2) holds, $f(x)\in B(A)$ and $\{f_k\}$ be a sequence of bounded real valued functions defined on a set A, then

(i) *If a sequence is pointwise lacunary statistical convergence of order α to the function f then it is pointwise $w_p^\alpha(f,\theta')$—summable to f,*
(ii) *If a sequence is pointwise lacunary statistical convergence of order α to the function f then it is pointwise $w_p(f,\theta')$—summable to f,*
(iii) *If a sequence is pointwise lacunary statistical convergence to the function f then it is pointwise $w_p(f,\theta')$—summable to f.*

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