Filomat 31:8 (2017), 2403–2412
DOI 10.2298/FIL1708403S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On $I-$ Lacunary Statistical Convergence of Order α of Sequences of Sets

Hacer Şengül[a], Mikail Et[b]

[a] *Department of Mathematics; Siirt University 56100; Siirt ; TURKEY*
[b] *Department of Mathematics; Fırat University 23119; Elazıg ; TURKEY*

Abstract. The idea of $I-$convergence of real sequences was introduced by Kostyrko et al. [*Kostyrko, P. ; Šalát, T. and Wilczyński, W. I−convergence, Real Anal. Exchange 26(2) (2000/2001), 669-686*] and also independently by Nuray and Ruckle [*Nuray, F. and Ruckle, W. H. Generalized statistical convergence and convergence free spaces, J. Math. Anal. Appl. 245(2) (2000), 513–527*]. In this paper we introduce the concepts of Wijsman $I-$lacunary statistical convergence of order α and Wijsman strongly $I-$lacunary statistical convergence of order α, and investigated between their relationship.

1. Introduction

The concept of statistical convergence was introduced by Steinhaus [36] and Fast [15]. Schoenberg [34] established some basic properties of statistical convergence and studied the concept as a summability method. Later on it was further investigated from the sequence space point of view and linked with summability theory by Altın et al. [1], Başarır and Konca [2], Caserta et al. [3], Connor [4], Çakallı [5], Çolak ([8],[9]), Et et al. ([11],[12],[20],[21]), Fridy [17], Gadjiev and Orhan [19], Kolk [22], Mursaleen et al. ([25],[26]), Salat [29], Savaş et al. ([10],[32],[33]) and many others. Nuray and Rhoades [28] extended the notion to statistical convergence of sequences of sets and gave some basic theorems. Ulusu and Nuray [38] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence.

Let X be a non-empty set. Then a family of sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I is additive *i.e.* $A, B \in I$ implies $A \cup B \in I$ and hereditary, *i.e.* $A \in I$, $B \subset A$ implies $B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a *filter* of X if and only if
(*i*) $\phi \notin F$, (*ii*) $A, B \in F$ implies $A \cap B \in F$ and (*iii*) $A \in F$, $A \subset B$ implies $B \in F$.

An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

A non-trivial ideal I is said to be *admissible* if $I \supset \{\{x\} : x \in X\}$.

If I is a non-trivial ideal in $X\left(X \neq \phi\right)$ then the family of sets
$F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter of X, called the *filter associated with* I.

Let (X, d) be a metric space. For any non-empty closed subset A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

Throughout the paper I will stand for a non-trivial admissible ideal of $\mathbb{N}$.

2010 *Mathematics Subject Classification*. 40A05, 40C05, 46A45
Keywords. $I-$convergence, Wijsman convergence, lacunary sequence
Received: 04 December 2015; Accepted: 25 April 2016
Communicated by Ljubiša D.R. Kočinac
Email addresses: hacer.sengul@hotmail.com (Hacer Şengül), mikailet@yahoo.com (Mikail Et)

The idea of $I-$convergence of real sequences was introduced by Kostyrko *et al.* [23] and also independently by Nuray and Ruckle [27] (who called it generalized statistical convergence) as a generalization of statistical convergence. Later on $I-$convergence was studied in ([6],[7],[14],[24],[30], [31],[32],[33],[37],[39]).

2. Main Results

In this section, we will extend the results of Et and Şengül ([13], [35]) to statistical convergence of set sequences, namely; the relationship between the concepts of Wijsman $I-$lacunary statistical convergence of order α and Wijsman strongly $I-$lacunary statistical convergence of order α are given

Definition 2.1. *Let (X,d) be a metric space, θ be a lacunary sequence, $\alpha \in (0,1]$ and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $I-$lacunary statistical convergent to A of order α (or $S_\theta^\alpha(I_w)-$convergent to A) if for each $\varepsilon > 0$, $\delta > 0$ and $x \in X$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \left| d(x, A_k) - d(x, A) \right| \geq \varepsilon \right\} \right| \geq \delta \right\}$$

belongs to I. In this case, we write $A_k \longrightarrow A \left(S_\theta^\alpha(I_w) \right)$. For $\theta = (2^r)$, we shall write $S^\alpha(I_w)$ instead of $S_\theta^\alpha(I_w)$ and in the special case $\alpha = 1$ and $\theta = (2^r)$ we shall write $S(I_w)$ instead of $S_\theta^\alpha(I_w)$.

As an example, consider the following sequence:

$$A_k = \begin{cases} \{3x\}, & k_{r-1} < k < k_{r-1} + \sqrt{h_r} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X$, $d(x,y) = \left| x - y \right|$, $A = \{1\}$, $x > 1$ and $\alpha = 1$. Since

$$\frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \left| d(x, A_k) - d(x, 1) \right| \geq \varepsilon \right\} \right| \geq \delta$$

belongs to I, the sequences $\{A_k\}$ is Wijsman $I-$lacunary statistical convergent to $\{1\}$ of order α ; that is $A_k \longrightarrow \{1\} \left(S_\theta^\alpha(I_w) \right)$.

Definition 2.2. *Let (X,d) be a metric space, θ be a lacunary sequence, $\alpha \in (0,1]$ and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly $I-$lacunary statistical convergent to A of order α (or $N_\theta^\alpha[I_w]-$convergent to A) if for each $\varepsilon > 0$ and $x \in X$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left| d(x, A_k) - d(x, A) \right| \geq \varepsilon \right\}$$

belongs to I. In this case, we write $A_k \longrightarrow A \left(N_\theta^\alpha[I_w] \right)$. For $\theta = (2^r)$, we shall write $N^\alpha[I_w]$ instead of $N_\theta^\alpha[I_w]$ and in the special case $\alpha = 1$ and $\theta = (2^r)$ we shall write $N[I_w]$ instead of $N_\theta^\alpha[I_w]$.

As an example, consider the following sequence:

$$A_k = \begin{cases} \left\{ \frac{xk}{2} \right\}, & k_{r-1} < k < k_{r-1} + \sqrt{h_r} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X$, $d(x,y) = \left| x - y \right|$, $A = \{1\}$, $x > 1$ and $\alpha = 1$. Since

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} \left| d(x, A_k) - d(x, 1) \right| \geq \varepsilon,$$

the sequences $\{A_k\}$ is Wijsman $I-$lacunary statistical convergent to $\{1\}$ of order α ; that is $A_k \longrightarrow \{1\} \left(N_\theta^\alpha[I_w] \right)$.

Theorem 2.3. *$S_\theta^\alpha(I_w) \cap L_\infty$ is a closed subset of L_∞ for $0 < \alpha \leq 1$.*

Proof. Omitted. □

Theorem 2.4. *Let (X, d) be a metric space, $\theta = (k_r)$ be a lacunary sequence and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X, then*

(i) $A_k \to A\left(N_\theta^\alpha[I_w]\right) \Rightarrow A_k \to A\left(S_\theta^\alpha(I_w)\right)$ and $N_\theta^\alpha[I_w]$ is a proper subset of $S_\theta^\alpha(I_w)$,

(ii) $\{A_k\} \in L_\infty$ and $A_k \to A\left(S_\theta^\alpha(I_w)\right) \Rightarrow A_k \to A\left(N_\theta^\alpha[I_w]\right)$,

(iii) $S_\theta^\alpha(I_w) \cap L_\infty = N_\theta^\alpha[I_w] \cap L_\infty$.

Proof. (i) The inclusion part of proof is easy. In order to show that the inclusion $N_\theta^\alpha[I_w] \subseteq S_\theta^\alpha(I_w)$ is proper, let θ be given and we define a sequence $\{A_k\}$ as follows

$$A_k = \begin{cases} \left\{x^2\right\}, & k = 1, 2, 3, ..., \left[\sqrt{h_r}\right] \\ \{0\}, & \text{otherwise} \end{cases}.$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X$, $d(x, y) = \left|x - y\right|$. We have for every $\varepsilon > 0, x > 0$ and $\frac{1}{2} < \alpha \leq 1$,

$$\frac{1}{h_r^\alpha}\left|\{k \in I_r : |d(x, A_k) - d(x, \{0\})| \geq \varepsilon\}\right| \leq \frac{\left[\sqrt{h_r}\right]}{h_r^\alpha},$$

and for any $\delta > 0$ we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\{k \in I_r : |d(x, A_k) - d(x, \{0\})| \geq \varepsilon\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{\left[\sqrt{h_r}\right]}{h_r^\alpha} \geq \delta\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to I, it follows that for $\frac{1}{2} < \alpha \leq 1$, $A_k \to \{0\}\left(S_\theta^\alpha(I_w)\right)$.

On the other hand, for $\frac{1}{2} < \alpha \leq 1$ and $x > 0$,

$$\frac{1}{h_r^\alpha}\sum_{k \in I_r}|d(x, A_k) - d(x, \{0\})| = \frac{\left(x^2 - 2x\right)\left[\sqrt{h_r}\right]}{h_r^\alpha} \to 0$$

and for $0 < \alpha < \frac{1}{2}$

$$\frac{\left(x^2 - 2x\right)\left[\sqrt{h_r}\right]}{h_r^\alpha} \to \infty.$$

Hence we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\sum_{k \in I_r}|d(x, A_k) - d(x, \{0\})| \geq 0\right\} = \left\{r \in \mathbb{N} : \frac{\left(x^2 - 2x\right)\left[\sqrt{h_r}\right]}{h_r^\alpha} \geq 0\right\} = \{a, a+1, a+2, ...\}$$

for some $a \in \mathbb{N}$ which belongs to $F(I)$, since I is admissible. So $A_k \nrightarrow \{0\}\left(N_\theta^\alpha[I_w]\right)$.

$ii)$ Suppose that $\{A_k\} \in L_\infty$ and $A_k \to A\left(S_\theta^\alpha(I_w)\right)$. Then we can assume that

$$|d(x, A_k) - d(x, A)| \leq M$$

for each $x \in X$ and all $k \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{h_r^\alpha}\sum_{k\in I_r}|d(x,A_k)-d(x,A)| &= \frac{1}{h_r^\alpha}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}|d(x,A_k)-d(x,A)| \\ &\quad+\frac{1}{h_r^\alpha}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|<\varepsilon}}|d(x,A_k)-d(x,A)| \\ &\leq \frac{M}{h_r^\alpha}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|+\varepsilon.\end{aligned}$$

Hence we have

$$\begin{aligned}&\left\{r\in\mathbb{N} : \frac{1}{h_r^\alpha}\sum_{k\in I_r}|d(x,A_k)-d(x,A)|\geq M\delta+\varepsilon\right\} \\ \subset\; &\left\{r\in\mathbb{N} : \frac{1}{h_r^\alpha}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\delta\right\}\in I.\end{aligned}$$

Therefore $A_k \to A\left(N_\theta^\alpha[I_w]\right)$.

iii) Follows from (*i*) and (*ii*). □

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence and α be a fixed real number such that $0 < \alpha \leq 1$. If $\liminf_r q_r > 1$, then $S^\alpha(I_w) \subset S_\theta^\alpha(I_w)$.*

Proof. Suppose first that $\liminf_r q_r > 1$; then there exists a $\lambda > 0$ such that $q_r \geq 1+\lambda$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r}\geq\frac{\lambda}{1+\lambda}\Longrightarrow\left(\frac{h_r}{k_r}\right)^\alpha\geq\left(\frac{\lambda}{1+\lambda}\right)^\alpha\Longrightarrow\frac{1}{k_r^\alpha}\geq\frac{\lambda^\alpha}{(1+\lambda)^\alpha}\frac{1}{h_r^\alpha}.$$

If $A_k \to A(S^\alpha(I_w))$, then for every $\varepsilon > 0$, for each $x \in X$, and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r^\alpha}\left|\{k\leq k_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right| &\geq \frac{1}{k_r^\alpha}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right| \\ &\geq \frac{\lambda^\alpha}{(1+\lambda)^\alpha}\frac{1}{h_r^\alpha}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|.\end{aligned}$$

For $\delta > 0$, we have

$$\begin{aligned}&\left\{r\in\mathbb{N} : \frac{1}{h_r^\alpha}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\delta\right\} \\ \subseteq\; &\left\{r\in\mathbb{N} : \frac{1}{k_r^\alpha}\left|\{k\leq k_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\frac{\delta\lambda^\alpha}{(1+\lambda)^\alpha}\right\}\in I.\end{aligned}$$

This completes the proof. □

Theorem 2.6. *Let $\theta = (k_r)$ be a lacunary sequence and the parameters α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $N_\theta^\alpha[I_w] \subseteq N_\theta^\beta[I_w]$ and the inclusion is strict.*

Proof. The inclusion part of proof is easy. To show that the inclusion is strict define $\{A_k\}$ such that for $(\mathbb{R}, d)$, $x > 1$ and $A = \{0\}$,

$$A_k=\begin{cases}\{3x+5\}, & k_{r-1}<k<k_{r-1}+\sqrt{h_r} \\ \{0\}, & \text{otherwise}\end{cases}.$$

Then $\{A_k\} \in N_\theta^\beta[I_w]$ for $\frac{1}{2} < \beta \leq 1$ but $\{A_k\} \notin N_\theta^\alpha[I_w]$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 2.7. *Let $\theta = (k_r)$ be a lacunary sequence and the parameters α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $S_\theta^\alpha (I_w) \subseteq S_\theta^\beta (I_w)$ and the inclusion is strict.*

Proof. The inclusion part of proof is easy. To show that the inclusion is strict define $\{A_k\}$ such that for $X = \mathbb{R}^2$

$$A_k = \begin{cases} (x,y) \in \mathbb{R}^2, x^2 + (y-1)^2 = k^2, & \text{if } k \text{ is square} \\ \{(0,0)\}, & \text{otherwise} \end{cases}.$$

Then $\{A_k\} \in S_\theta^\beta (I_w)$ for $\frac{1}{2} < \beta \leq 1$ but $\{A_k\} \notin S_\theta^\alpha (I_w)$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 2.8. *Let the parameters α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $S^\beta (I_w) \subseteq N^\alpha [I_w]$.*

Proof. For any sequence $\{A_k\}$ and $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{n^\alpha} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| &\geq \frac{1}{n^\alpha} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \, \varepsilon \\ &\geq \frac{1}{n^\beta} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \, \varepsilon \end{aligned}$$

and so

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n^\beta} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \frac{\delta}{\varepsilon}\right\} \in I.$$

This gives that $S^\beta (I_w) \subseteq N^\alpha [I_w]$. □

Theorem 2.9. *Let $\theta = (k_r)$ be a lacunary sequence and α be a fixed real number such that $0 < \alpha \leq 1$. If $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{k_r} > 0$ then $S(I_w) \subseteq S_\theta^\alpha (I_w)$.*

Proof. Let (X, d) be a metric space, $\theta = (k_r)$ be a lacunary sequence and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. If $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{k_r} > 0$, then we can write

$$\begin{aligned} \{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\} &\supset \{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \\ \frac{1}{k_r} |\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| &\geq \frac{1}{k_r} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ &= \frac{h_r^\alpha}{k_r} \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|. \end{aligned}$$

So

$$\begin{aligned} &\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\} \\ \subseteq &\left\{r \in \mathbb{N} : \frac{1}{k_r} |\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta \frac{h_r^\alpha}{k_r}\right\} \end{aligned}$$

which implies that $S(I_w) \subseteq S_\theta^\alpha (I_w)$. □

Theorem 2.10. *Let (X, d) be a metric space and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. If $\theta = (k_r)$ is a lacunary sequence with $\lim\sup \frac{(k_j - k_{j-1})^\alpha}{k_{r-1}^\alpha} < \infty$ $(j = 1, 2, ..., r)$, then $A_k \to A\left(S_\theta^\alpha (I_w)\right)$ implies $A_k \to A(S^\alpha (I_w))$.*

Proof. If $\limsup \frac{\left(k_j-k_{j-1}\right)^{\alpha}}{k_{r-1}^{\alpha}} < \infty$, then without any loss of generality, we can assume that there exists a $0 < B_j < \infty$ such that $\frac{\left(k_j-k_{j-1}\right)^{\alpha}}{k_{r-1}^{\alpha}} < B_j$, $(j = 1, 2, ..., r)$ for all $r \geq 1$. Suppose that $A_k \to A\left(S_\theta^\alpha (I_w)\right)$ and for $\varepsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| < \delta \right\}$$

and

$$T = \left\{ r \in \mathbb{N} : \frac{1}{n^\alpha} \left|\left\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| < \delta_1 \right\}.$$

It is obvious from our assumption that $C \in F(I)$, the filter associated with the ideal I. Further observe that

$$A_i = \frac{1}{h_i^\alpha} \left|\left\{k \in I_i : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| < \delta$$

for all $i \in C$. Let $n \in N$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned} \frac{1}{n^\alpha} \left|\left\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| &\leq \frac{1}{k_{r-1}^\alpha} \left|\left\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| \\ &= \frac{1}{k_{r-1}^\alpha} \left|\left\{k \in I_1 : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| + ... \\ &\quad + \frac{1}{k_{r-1}^\alpha} \left|\left\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| \\ &= \frac{k_1^\alpha}{k_{r-1}^\alpha} \frac{1}{h_1^\alpha} \left|\left\{k \in I_1 : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| \\ &\quad + \frac{(k_2 - k_1)^\alpha}{k_{r-1}^\alpha} \frac{1}{h_2^\alpha} \left|\left\{k \in I_2 : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| \\ &\quad + ... + \frac{(k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}\right| \\ &\leq \sup_{i \in C} A_i . \frac{k_1^\alpha + (k_2 - k_1)^\alpha + ... + (k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} \\ &\leq \sup_{i \in C} A_i \left(B_1 + B_2 + ... + B_r\right) < \delta \sum_{j=1}^{r} B_j. \end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{\sum_{j=1}^{r} B_j}$ and in view of the fact that $\cup \{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(I)$. This completes the proof of the theorem. □

In [16], it is defined that the lacunary sequence $\theta' = (s_r)$ is called a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subseteq (s_r)$. In [18], the inclusion relationship between S_θ and $S_{\theta'}$ is studied.

Theorem 2.11. *Suppose $\theta' = (s_r)$ is a lacunary refinement of the lacunary sequence $\theta = (k_r)$. Let $I_r = (k_{r-1}, k_r]$ and $J_r = (s_{r-1}, s_r]$, $r = 1, 2, 3, ...$ If there exists $\epsilon > 0$ such that for $0 < \alpha \leq \beta \leq 1$,*

$$\frac{\left|J_j\right|^\beta}{|I_i|^\alpha} \geq \epsilon \text{ for every } J_j \subseteq I_i .$$

Then $A_k \to A\left(S_\theta^\alpha (I_w)\right)$ implies $A_k \to A\left(S_{\theta'}^\beta (I_w)\right)$, i.e., $S_\theta^\alpha (I_w) \subseteq S_{\theta'}^\beta (I_w)$.

Proof. For any $\varepsilon > 0$, and every J_j, we can find I_i such that $J_j \subseteq I_i$; then we have

$$\begin{aligned}\frac{1}{\left|J_j\right|^{\beta}}\left|\left\{k \in J_j : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| &= \left(\frac{|I_i|^{\alpha}}{\left|J_j\right|^{\beta}}\right)\left(\frac{1}{|I_i|^{\alpha}}\right)\left|\left\{k \in J_j : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| \\ &\leq \left(\frac{|I_i|^{\alpha}}{\left|J_j\right|^{\beta}}\right)\left(\frac{1}{|I_i|^{\alpha}}\right)\left|\left\{k \in I_i : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| \\ &\leq \left(\frac{1}{\epsilon}\right)\left(\frac{1}{|I_i|^{\alpha}}\right)\left|\left\{k \in I_i : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right|,\end{aligned}$$

and so

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{\left|J_j\right|^{\beta}}\left|\left\{k \in J_j : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ \subseteq\ &\left\{r \in \mathbb{N} : \left(\frac{1}{|I_i|^{\alpha}}\right)\left|\left\{k \in I_i : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| \geq \delta\epsilon\right\} \in I.\end{aligned}$$

The proof completes immediately. □

Theorem 2.12. *Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are two lacunary sequences. Let $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $r = 1,2,3,...$, and $I_{ij} = I_i \cap J_j$, $i,j = 1,2,3,....$ If there exists $\epsilon > 0$ such that for $0 < \alpha \leq \beta \leq 1$,*

$$\frac{\left|I_{ij}\right|^{\beta}}{|I_i|^{\alpha}} \geq \epsilon \text{ for every } i,j = 1,2,3,..., \text{provided } I_{ij} \neq \varnothing.$$

Then $A_k \to A\left(S_{\theta}^{\alpha}(I_w)\right)$ implies $A_k \to A\left(S_{\theta'}^{\beta}(I_w)\right)$, i.e., $S_{\theta}^{\alpha}(I_w) \subseteq S_{\theta'}^{\beta}(I_w)$.

Proof. Let $\theta'' = \theta' \cup \theta$. Then θ'' is a lacunary refinement of the lacunary sequence θ', also θ. Then interval sequence of θ'' is $\left\{I_{ij} = I_i \cap J_j : I_{ij} \neq \varnothing\right\}$. From Theorem 2.11, the condition in Theorem 2.12: $\frac{\left|I_{ij}\right|^{\beta}}{|I_i|^{\alpha}} \geq \epsilon$, for every $i,j = 1,2,3,...$,provided $I_{ij} \neq \varnothing$ yields that $A_k \to A\left(S_{\theta}^{\alpha}(I_w)\right)$implies $A_k \to A\left(S_{\theta''}^{\beta}(I_w)\right)$. Since θ'' is also a lacunary refinement of the lacunary sequence θ', we have that $A_k \to A\left(S_{\theta''}^{\alpha}(I_w)\right)$implies $A_k \to A\left(S_{\theta'}^{\beta}(I_w)\right)$. The proof follows immediately. □

Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$. Now we shall give some general inclusion relations between the sets of $S_{\theta}^{\alpha}(I_w)$ −convergent sequences and $N_{\theta}^{\alpha}[I_w]$ −summable sequences for different $\alpha's$ and $\theta's$ which also include Theorem 2.4, Theorem 2.6, Theorem 2.7 and Theorem 2.8 as a special case.

Theorem 2.13. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be such that $0 < \alpha \leq \beta \leq 1$,*

(i) If

$$\lim_{r\to\infty}\inf \frac{h_r^{\alpha}}{\ell_r^{\beta}} > 0 \tag{1}$$

then $S_{\theta'}^{\beta}(I_w) \subseteq S_{\theta}^{\alpha}(I_w)$,

(ii) If

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^{\beta}}=1 \tag{2}$$

then $S_{\theta}^{\alpha}(I_w)\subseteq S_{\theta'}^{\beta}(I_w)$.

Proof. (i) Let (X,d) be a metric space, $\theta=(k_r)$ be a lacunary sequence and A, A_k (for all $k\in\mathbb{N}$) be non-empty closed subsets of X. Suppose that $I_r\subset J_r$ for all $r\in\mathbb{N}$ and let (1) be satisfied. For given $\varepsilon>0$ we have

$$\{k\in J_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\supseteq\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\},$$

and so

$$\frac{1}{\ell_r^{\beta}}\left|\{k\in J_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\frac{h_r^{\alpha}}{\ell_r^{\beta}}\frac{1}{h_r^{\alpha}}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|.$$

Hence

$$\begin{aligned}&\left\{r\in\mathbb{N} : \frac{1}{h_r^{\alpha}}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\delta\right\}\\ \subseteq\ &\left\{r\in\mathbb{N} : \frac{1}{\ell_r^{\beta}}\left|\{k\in J_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\delta\frac{h_r^{\alpha}}{\ell_r^{\beta}}\right\}\in I\end{aligned}$$

for all $r\in\mathbb{N}$, where $I_r=(k_{r-1},k_r]$, $J_r=(s_{r-1},s_r]$, $h_r=k_r-k_{r-1}$, $\ell_r=s_r-s_{r-1}$. Now taking the limit as $r\to\infty$ in the last inequality and using (1) we get $S_{\theta'}^{\beta}(I_w)\subseteq S_{\theta}^{\alpha}(I_w)$.

(ii) Omitted. □

Theorem 2.14. *Let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subseteq J_r$ for all $r\in\mathbb{N}$, α and β be fixed real numbers such that $0<\alpha\leq\beta\leq1$. Then we have*

(i) *If* (1) *holds then* $N_{\theta'}^{\beta}[I_w]\subset N_{\theta}^{\alpha}[I_w]$,

(ii) *If* (2) *holds and* $\{A_k\}\in L_\infty$ *then* $N_{\theta}^{\alpha}[I_w]\subset N_{\theta'}^{\beta}[I_w]$.

Proof. Omitted. □

Theorem 2.15. *Let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subseteq J_r$ for all $r\in\mathbb{N}$, α and β be fixed real numbers such that $0<\alpha\leq\beta\leq1$. Then*

(i) *Let* (1) *holds, if a sequence is strongly* $N_{\theta'}^{\beta}[I_w]$ *−summable to* A, *then it is* $S_{\theta}^{\alpha}(I_w)$ *−statistically convergent to* A,

(ii) *Let* (2) *holds and* $\{A_k\}$ *be a bounded sequence, if a sequence is* $S_{\theta}^{\alpha}(I_w)$ *−statistically convergent to* A *then it is strongly* $N_{\theta'}^{\beta}[I_w]$ *−summable to* A.

Proof. (i) Omitted.

(ii) Suppose that $S_\theta^\alpha(I_w)-\lim A_k=A$ and $\{A_k\}\in L_\infty$. Then there exists some $M>0$ such that $|d(x,A_k)-d(x,A)|\leq M$ for all k, then for every $\varepsilon>0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^\beta}\sum_{k\in J_r}|d(x,A_k)-d(x,A)| &= \frac{1}{\ell_r^\beta}\sum_{k\in J_r-I_r}|d(x,A_k)-d(x,A)|+\frac{1}{\ell_r^\beta}\sum_{k\in I_r}|d(x,A_k)-d(x,A)|\\
&\leq \left(\frac{\ell_r-h_r}{\ell_r^\beta}\right)M+\frac{1}{\ell_r^\beta}\sum_{k\in I_r}|d(x,A_k)-d(x,A)|\\
&\leq \left(\frac{\ell_r-h_r^\beta}{\ell_r^\beta}\right)M+\frac{1}{\ell_r^\beta}\sum_{k\in I_r}|d(x,A_k)-d(x,A)|\\
&\leq \left(\frac{\ell_r}{h_r^\beta}-1\right)M+\frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}|d(x,A_k)-d(x,A)|\\
&\quad+\frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|<\varepsilon}}|d(x,A_k)-d(x,A)|\\
&\leq \left(\frac{\ell_r}{h_r^\beta}-1\right)M+\frac{M}{h_r^\alpha}\left|\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|+\frac{\ell_r}{h_r^\beta}\varepsilon
\end{aligned}$$

and so

$$\begin{aligned}
&\left\{r\in\mathbb{N}:\frac{1}{\ell_r^\beta}\sum_{k\in J_r}|d(x,A_k)-d(x,A)|\geq\delta\right\}\\
\subseteq\ &\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\left|\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\frac{\delta}{M}\right\}\in I,
\end{aligned}$$

for all $r\in\mathbb{N}$. Using (2) we obtain that $N_{\theta'}^\beta[I_w]-\lim A_k=A$, whenever $S_\theta^\alpha(I_w)-\lim A_k=A$. □

References

[1] Y. Altın, M. Et, M. Başarır, On some generalized difference sequences of fuzzy numbers, Kuwait J. Sci. Engrg. 34(1A) (2007) 1–14.
[2] M. Başarır, Ş. Konca, On some spaces of lacunary convergent sequences derived by Nörlund-type mean and weighted lacunary statistical convergence, Arab J. Math. Sci. 20(2) (2014) 250–263.
[3] A. Caserta, G. Di Maio, and L. D. R. Kočinac, Statistical convergence in function spaces, Abstr. Appl. Anal., Art. ID 420419 (2011) 11 pp.
[4] J. S. Connor, The Statistical and strong $p-$Cesàro convergence of sequences, Analysis 8 (1988) 47-63.
[5] H. Çakallı, Lacunary statistical convergence in topological groups, Indian J. Pure Appl. Math. 26(2) (1995) 113–119.
[6] H. Çakallı, B. Hazarika, Ideal quasi-Cauchy sequences, J. Inequal. Appl. 2012:234 (2012) 11 pp.
[7] H. Çakallı, A variation on ward continuity, Filomat 27(8) (2013) 1545–1549.
[8] R. Çolak, Statistical convergence of order α Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub, (2010) 121–129.
[9] R. Çolak, On λ-Statisitical Convergence, Conference on Summability and Applications 2011 Istanbul Commerce Univ. May 12-13 (2011) İstanbul.
[10] P. Das, E. Savaş, S. Kr. Ghosal, On generalizations of certain summability methods using ideals, Appl. Math. Lett. 24 (9) (2011) 1509–1514.
[11] M. Et, Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences, Appl. Math. Comput. 219(17) (2013) 9372–9376.
[12] M. Et, Strongly almost summable difference sequences of order m defined by a modulus, Studia Sci. Math. Hungar. 40(4) (2003) 463–476.
[13] M. Et, H. Şengül, Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α, Filomat 28(8) (2014) 1593–1602.
[14] M. Et, A. Alotaibi, S. A. Mohiuddine, On (Δ^m, I) −statistical convergence of order α,The Scientific World Journal Volume, Article ID 535419 (2014) 5 pages.

[15] H. Fast, Sur la convergence statistique. Colloq. Math. 2 (1951) 241-244.
[16] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesàro-type summability spaces, Proc. London Math. Soc. 37(3) (1978) 508–520.
[17] J. A. Fridy, On statistical convergence. Analysis 5 (1985) 301-313.
[18] J. A. Fridy, C. Orhan, Lacunary statistical convergence. Pacific J. Math. 160(1) (1993) 43–51.
[19] A. D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32(1) (2002) 129–138.
[20] A. Gökhan, M. Et, M. Mursaleen, Almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers, Math. Comput. Modelling 49(3-4) (2009) 548–555.
[21] M. Güngör, M. Et, Δ^r-strongly almost summable sequences defined by Orlicz functions, Indian J. Pure Appl. Math. 34(8) (2003) 1141–1151.
[22] E. Kolk, The statistical convergence in Banach spaces, Acta Comment. Univ. Tartu 928 (1991) 41-52.
[23] P. Kostyrko, T. Šalát, W. Wilczyński, $I-$convergence, Real Anal. Exchange 26 (2000/2001) 669–686.
[24] P. Kostyrko, M. Mačaj, M. Sleziak, T. Šalát, $I-$convergence and extremal $I-$limit points, Math. Slovaca 55(4) (2005) 443–464.
[25] M. Mursaleen, V. Karakaya, M. Ertürk, F. Gürsoy, Weighted statistical convergence and its application to Korovkin type approximation theorem, Appl. Math. Comput. 218(18) (2012) 9132–9137.
[26] M.Mursaleen, and S. A. Mohiuddine, Korovkin type approximation theorem for almost and statistical convergence, Nonlinear analysis, 487–494, Springer Optim. Appl., 68 (2012) Springer, New York.
[27] F. Nuray, W. H. Ruckle, Generalized statistical convergence and convergence free spaces, J. Math. Anal. Appl. 245(2) (2000) 513–527.
[28] F. Nuray, B. E. Rhoades, Statistical convergence of sequences of sets, Fasc. Math. 49 (2012) 87–99.
[29] T. Šalát, On statistically convergent sequences of real numbers, Math. Slovaca 30 (1980) 139–150.
[30] T. Šalát, B. C. Tripathy, M. Ziman, On $I-$convergence field, Ital. J. Pure Appl. Math. No. 17 (2005) 45–54.
[31] T. Šalát, B. C. Tripathy, M. Ziman, On some properties of $I-$convergence, Tatra Mt. Math. Publ. 28 part II (2004) 279–286.
[32] E. Savaş, P. A. Das, Generalized statistical convergence via ideals, Appl. Math. Lett. 24(6) (2011) 826–830.
[33] E. Savaş, On I-lacunary statistical convergence of order α for sequences of sets, Filomat 29(6) (2015) 1223–1229.
[34] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.
[35] H. Şengül, M. Et, On lacunary statistical convergence of order α, Acta Math. Sci. Ser. B Engl. Ed. 34 (2) (2014) 473–482.
[36] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73–74.
[37] B. C. Tripathy, B. Hazarika, Paranorm $I-$convergent sequence spaces, Math. Slovaca 59(4) (2009) 485–494.
[38] U. Ulusu, F. Nuray, Lacunary statistical convergence of sequence of sets, Prog. Appl. Math. 4(2) (2012) 99-109.
[39] U. Ulusu, E. Dündar, $I-$lacunary statistical convergence of sequences of sets, Filomat 28(8) (2014) 1567–1574.