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ON WIJSMAN $\mathcal{I}-$ LACUNARY STATISTICAL EQUIVALENCE OF ORDER (η, μ)

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Abstract. The idea of asymptotically equivalent sequences and asymptotic regular matrices was introduced by Marouf [*Marouf, M. Asymptotic equivalence and summability, Int. J. Math. Sci. 16(4) 755-762 (1993)*] and Patterson [*Patterson, RF. On asymptotically statistically equivalent sequences, Demonstr. Math. 36(1), 149-153 (2003)*] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In this paper we introduce the concepts of Wijsman asymptotically $\mathcal{I}-$lacunary statistical equivalence of order (η, μ) and strongly asymptotically $\mathcal{I}-$lacunary equivalence of order (η, μ) of sequences of sets and investigated between their relationship.

1. Introduction

The concept of statistical convergence was introduced by Fast [10] and Steinhaus [23] and later reintroduced by Schoenberg [22]. Later on it was further investigated from the sequence space point of view and linked with summability theory by Çakallı [4], Çolak [5], Connor [3], Et et al. ([6], [7], [8], [9]), Altınok et al. [1], Işık and Altın [15], Işık and Akbaş [14], Fridy [12], Salat [24], Belen et al. [2], Şengül ([31], [32]), Şengül and Et [33], Ulusu and Savaş [37] and many others. Nuray and Rhoades [19] extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems. Ulusu and Nuray ([35], [36]) defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence.

Let X be non-empty set. Then a family of sets $\mathcal{I} \subseteq 2^X$ (power sets of X) is said to be an *ideal* if $\mathcal{I}$ is additive *i.e.* $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and hereditary, *i.e.* $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

A non-empty family of sets $\mathcal{F} \subseteq 2^X$ is said to be a *filter* of X iff

(i) $\emptyset \notin \mathcal{F}$,

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,

(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

An ideal $\mathcal{I} \subseteq 2^X$ is called *non-trivial* if $\mathcal{I} \neq 2^X$.

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A non-trivial ideal $\mathcal{I}$ is said to be *admissible* if $\mathcal{I}\supset\{\{x\}:x\in X\}$.

If $\mathcal{I}$ is a non-trivial ideal in $X(X\neq\emptyset)$ then the family of sets $\mathcal{F}(\mathcal{I})=\{M\subset X:(\exists A\in\mathcal{I})(M=X\setminus A)\}$ is a filter of X, called the *filter associated with* $\mathcal{I}$.

Let (X,ρ) be a metric space. For any point $x\in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x,A)=\inf_{a\in A}\rho(x,a).$$

Let (X,ρ) be a metric space. For any non-empty closed subsets A, $A_k\subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k\to\infty}d(x,A_k)=d(x,A)$$

for each $x\in X$. In this case we write $W-\lim A_k=A$.

Throughout the paper $\mathcal{I}$ will stand for a non-trivial admissible ideal of $\mathbb{N}$.

Let (X,ρ) be a metric space. For any non-empty closed subset A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x,A_k)<\infty$ for each $x\in X$. In this case we write $\{A_k\}\in L_\infty$.

The idea of $\mathcal{I}-$convergence of real sequences was introduced by Kostyrko *et al.* [16] and also independently by Nuray and Ruckle [20] (who called it generalized statistical convergence) as a generalization of statistical convergence. Later on $\mathcal{I}-$convergence was studied in ([9], [30], [17], [25], [26], [27], [28], [29], [34], [38]).

2. Main Results

Marouf [18] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Patterson [21] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

In this part, we investigate the relation between the concepts of Wijsman asymptotically $\mathcal{I}-$lacunary statistical equivalence of order (η,μ) and strongly asymptotically $\mathcal{I}-$lacunary equivalence of order (η,μ) for $0<\eta\leq\mu\leq 1$.

Definition 2.1. *Let (X,ρ) be a metric space, $0<\eta\leq\mu\leq1$ and $\mathcal{I}\subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A_k,B_k\subset X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically $\mathcal{I}-$statistical equivalent of order (η,μ) (Wijsman sense) of multiple L if for every $\varepsilon>0,\delta>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n^\eta}\left|\left\{k\leq n:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu\geq\delta\right\}\in\mathcal{I}.$$

In the present case, we write $A_k\overset{WS_\eta^\mu(\mathcal{I})}{\sim}B_k$.

Definition 2.2. *Let (X,ρ) be a metric space, θ be a lacunary sequence, $0<\eta\leq\mu\leq1$ and $\mathcal{I}\subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A_k,B_k\subset X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically $\mathcal{I}-$lacunary statistical equivalent of order (η,μ) (Wijsman sense) of multiple L if for every $\varepsilon>0,\delta>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\eta}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu\geq\delta\right\}\in\mathcal{I}.$$

In the present case, we write $A_k \overset{WS_{\eta}^{\mu}(\theta,\mathcal{I})}{\sim} B_k$.

As an example, consider

$$A_k = \begin{cases} \{x - 2yk^2\}, & \text{if } k \text{ is a square integer} \\ \{2x\}, & \text{otherwise} \end{cases},$$

$$B_k = \begin{cases} \{x - 3yk^2\}, & \text{if } k \text{ is a square integer} \\ \{2x\}, & \text{otherwise} \end{cases},$$

sequences and let $(\mathbb{R}, \rho)$ be a metric space such that for $x, y \in X$, $d(x,y) = |x-y|$ and $L = 1$. Since

$$\frac{1}{h_r^{\eta}} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - 1 \right| \geq \varepsilon \right\} \right|^{\mu} \geq \delta$$

for $\eta = \frac{1}{2}$ and $\mu = \frac{3}{5}$, the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically $\mathcal{I}-$lacunary statistical equivalent of order (η, μ) (Wijsman sense) ; that is $A_k \overset{WS_{\eta}^{\mu}(\theta,\mathcal{I})}{\sim} B_k$.

Definition 2.3. *Let (X, ρ) be a metric space, θ be a lacunary sequence, $0 < \eta \leq \mu \leq 1$ and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically $\mathcal{I}-$lacunary equivalent of order (η, μ) (Wijsman sense) of multiple L if for every $\varepsilon > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^{\eta}} \left(\sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \right)^{\mu} \geq \varepsilon \right\} \in \mathcal{I}.$$

In the present case, we write $A_k \overset{WN_{\eta}^{\mu}[\theta,\mathcal{I}]}{\sim} B_k$.

As an example, consider the following sequences

$$A_k = \begin{cases} \{x - 2y^2k\}, & \text{if } k \text{ is a square integer} \\ \{\frac{x}{2}\}, & \text{otherwise} \end{cases},$$

$$B_k = \begin{cases} \{x - 5y^2k\}, & \text{if } k \text{ is a square integer} \\ \{\frac{x}{2}\}, & \text{otherwise} \end{cases},$$

and let $(\mathbb{R}, \rho)$ be a metric space such that for $x, y \in X$, $d(x,y) = |x-y|$, $L = 1$. Since

$$\frac{1}{h_r^{\eta}} \left(\sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - 1 \right| \right)^{\mu} \geq \varepsilon$$

for $\eta = \frac{1}{2}$ and $\mu = \frac{4}{5}$, the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically $\mathcal{I}-$lacunary equivalent of order (η, μ) (Wijsman sense); that is $A_k \overset{WN_{\eta}^{\mu}[\theta,\mathcal{I}]}{\sim} B_k$.

Theorem 2.1. *Let (X, ρ) be a metric space, $0 < \eta \leq \mu \leq 1$, $\theta = \{k_r\}$ be a lacunary sequence and let A_k, B_k be non-empty closed subsets of X.*

i) If $A_k \overset{WN_{\eta}^{\mu}[\theta,\mathcal{I}]}{\sim} B_k$, then $A_k \overset{WS_{\eta}^{\mu}(\theta,\mathcal{I})}{\sim} B_k$,

ii) If $A_k \overset{WS_{\eta}^{\mu}(\theta,\mathcal{I})}{\sim} B_k$ and $\{A_k\} \in L_{\infty}$, then $A_k \overset{WN_{\eta}^{\mu}[\theta,\mathcal{I}]}{\sim} B_k$ for $\eta = \mu$.

Proof. Omitted. □

Theorem 2.2. *Let* $0<\eta\leq\mu\leq 1$. *If* $\theta=\{k_r\}$ *is a lacunary sequence with* $\liminf_r q_r>1$, *then* $A_k \overset{WS_\eta^\mu(\mathcal{I})}{\sim} B_k$ *implies* $A_k \overset{WS_\eta^\mu(\theta,\mathcal{I})}{\sim} B_k$.

Proof. Let A_k, B_k be non-empty closed subsets of X. Suppose first that $\liminf_r q_r> 1$; then we have $q_r\geq 1+\lambda$ for $\lambda>0$ and sufficiently large r. So we can write

$$\frac{h_r}{k_r}\geq\frac{\lambda}{1+\lambda}\Longrightarrow\left(\frac{h_r}{k_r}\right)^\eta\geq\left(\frac{\lambda}{1+\lambda}\right)^\eta\Longrightarrow\frac{1}{k_r^\eta}\geq\frac{\lambda^\eta}{(1+\lambda)^\eta}\frac{1}{h_r^\eta}.$$

If $A_k \overset{WS_\eta^\mu(\mathcal{I})}{\sim} B_k$, then for every $\varepsilon>0$ and each $x\in X$, we have

$$\begin{aligned}\frac{1}{k_r^\eta}\left|\left\{k\leq k_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu &\geq \frac{1}{k_r^\eta}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu\\ &\geq \frac{\lambda^\eta}{(1+\lambda)^\eta}\frac{1}{h_r^\eta}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu.\end{aligned}$$

For $\delta>0$, we have

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{h_r^\eta}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu\geq\delta\right\}\\ \subseteq\ &\left\{r\in\mathbb{N}:\frac{1}{k_r^\eta}\left|\left\{k\leq k_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu\geq\frac{\delta\lambda^\eta}{(1+\lambda)^\eta}\right\}\in\mathcal{I}.\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let* $0<\eta\leq\mu\leq 1$. *If* $\lim_{r\to\infty}\inf\frac{h_r^\eta}{k_r}>0$ *then* $A_k \overset{WS(\mathcal{I})}{\sim} B_k$ *implies* $A_k \overset{WS_\eta^\mu(\theta,\mathcal{I})}{\sim} B_k$.

Proof. Let (X,ρ) be a metric space and A_k, B_k be non-empty closed subsets of X. If $\lim_{r\to\infty}\inf\frac{h_r^\eta}{k_r}>0$, then

$$\begin{aligned}\left\{k\leq k_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\} &\supset \left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\\ \frac{1}{k_r}\left|\left\{k\leq k_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right| &\geq \frac{1}{k_r}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu\\ &= \frac{h_r^\eta}{k_r}\frac{1}{h_r^\eta}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu.\end{aligned}$$

So

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{h_r^\eta}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^\mu\geq\delta\right\}\\ \subseteq\ &\left\{r\in\mathbb{N}:\frac{1}{k_r}\left|\left\{k\leq k_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\frac{h_r^\eta}{k_r}\right\}\in\mathcal{I}\end{aligned}$$

which implies that $A_k \overset{WS_\eta^\mu(\theta,\mathcal{I})}{\sim} B_k$. □

Theorem 2.4. *Let* (X,ρ) *be a metric space and* A_k, B_k *be non-empty closed subsets of* X. *If* $\theta=\{k_r\}$ *is a lacunary sequence with* $\limsup\frac{(k_j-k_{j-1})^\eta}{k_{r-1}^\eta}<\infty$ $(j=1,2,...,r)$, *then* $A_k \overset{WS_\eta^\mu(\theta,\mathcal{I})}{\sim} B_k$ *implies* $A_k \overset{WS_\eta^\mu(\mathcal{I})}{\sim} B_k$.

Proof. If $\limsup \frac{(k_j-k_{j-1})^\eta}{k_{r-1}^\eta} < \infty$, then there exists a $0 < B_j < \infty$ such that $\frac{(k_j-k_{j-1})^\eta}{k_{r-1}^\eta} < B_j$, $(j = 1, 2, ..., r)$ for all $r \geq 1$. Suppose that $A_k \overset{WS_\eta^\mu(\theta,\mathcal{I})}{\sim} B_k$ and for $\varepsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{ r \in \mathbb{N} : \frac{1}{h_r^\eta} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu < \delta \right\} \in \mathcal{F}(\mathcal{I})$$

and

$$T = \left\{ n \in \mathbb{N} : \frac{1}{n^\eta} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu < \delta_1 \right\}.$$

Further we can write

$$a_i = \frac{1}{h_i^\eta} \left| \left\{ k \in I_i : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu < \delta$$

for all $i \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
\frac{1}{n^\eta} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu &\leq \frac{1}{k_{r-1}^\eta} \left| \left\{ k \leq k_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu \\
&\leq \frac{1}{k_{r-1}^\eta} \left| \left\{ k \in I_1 : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu + ... \\
&\quad + \frac{1}{k_{r-1}^\eta} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu \\
&= \frac{k_1^\eta}{k_{r-1}^\eta} \frac{1}{h_1^\eta} \left| \left\{ k \in I_1 : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu \\
&\quad + \frac{(k_2-k_1)^\eta}{k_{r-1}^\eta} \frac{1}{h_2^\eta} \left| \left\{ k \in I_2 : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu \\
&\quad + ... + \frac{(k_r-k_{r-1})^\eta}{k_{r-1}^\eta} \frac{1}{h_r^\eta} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|^\mu \\
&\leq \sup_{i \in C} a_i . \frac{k_1^\eta + (k_2-k_1)^\eta + ... + (k_r-k_{r-1})^\eta}{k_{r-1}^\eta} \\
&\leq \sup_{i \in C} a_i \left(B_1 + B_2 + ... + B_r \right) < \delta \sum_{j=1}^r B_j.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{\sum_{j=1}^r B_j}$ and in view of the fact that $\cup \{n : k_{r-1} < n < k_r\ ,\ r \in C\} \subset T$ where $C \in \mathcal{F}(\mathcal{I})$. Thus $T \in \mathcal{F}(\mathcal{I})$ is obtained. □

Theorem 2.5. *Let A_k, B_k be non-empty closed subsets of X and η_1, η_2, μ_1 and μ_2 be positive real numbers such that $0 < \eta_1 \leq \eta_2 \leq \mu_1 \leq \mu_2 \leq 1$, then $A_k \overset{WN_{\eta_1}^{\mu_2}[\theta,\mathcal{I}]}{\sim} B_k$ implies $A_k \overset{WN_{\eta_2}^{\mu_1}[\theta,\mathcal{I}]}{\sim} B_k$, but the converse doesn't hold.*

Proof. The first part of the proof is easy and so omitted. To show the converse; define two sequences $\{A_k\}$ and $\{B_k\}$ and consider metric space $(\mathbb{R}, \rho)$ such that for $x > 1$,

$$A_k = \begin{cases} \left\{x^2 + x - 2\right\}, & \text{if } k \text{ is square} \\ \{x\}, & \text{otherwise} \end{cases}$$

$$B_k=\begin{cases}\{1\}, & \text{if } k \text{ is square}\\ \{0\}, & \text{otherwise}\end{cases}.$$

Then $A_k \overset{WN_{\eta_2}^{\mu_1}[\theta,\mathcal{I}]}{\sim} B_k$ for $\eta_2=\mu_1=\frac{1}{2}$ but $A_k \overset{WN_{\eta_1}^{\mu_2}[\theta,\mathcal{I}]}{\nsim} B_k$ for $\eta_1=\frac{1}{4}$, $\mu_2=1$, and $L=0$. □

The following result is a consequence of Theorem 2.5.

Corollary 2.6. *Let (X,ρ) be a metric space, $\theta=\{k_r\}$ be a lacunary sequence, $0<\eta_1\leq\eta_2\leq\mu_1\leq\mu_2\leq 1$ and A_k, B_k be non-empty closed subsets of X.*
(i) If $A_k \overset{WN_{\eta_1}[\theta,\mathcal{I}]}{\sim} B_k$ implies $A_k \overset{WN_{\eta_2}[\theta,\mathcal{I}]}{\sim} B_k$ for $\mu_1=\mu_2=1$.
(ii) If $A_k \overset{WN_{\eta_1}[\theta,\mathcal{I}]}{\sim} B_k$ implies $A_k \overset{WN[\theta,\mathcal{I}]}{\sim} B_k$ for $\eta_2=\mu_1=\mu_2=1$.

Theorem 2.7. *Let η_1,η_2,μ_1 and μ_2 be positive real numbers such that $0<\eta_1\leq\eta_2\leq\mu_1\leq\mu_2\leq 1$, then $A_k \overset{WS_{\eta_1}^{\mu_2}(\theta,\mathcal{I})}{\sim} B_k$ implies $A_k \overset{WS_{\eta_2}^{\mu_1}(\theta,\mathcal{I})}{\sim} B_k$, but the converse doesn't hold.*

Proof. The first part of the proof is easy and so omitted. To show the converse; define two sequences $\{A_k\}$ and $\{B_k\}$ and consider metric space $(\mathbb{R},\rho)$ such that for $x>1$,

$$A_k=\begin{cases}\left\{(x,y)\in\mathbb{R}^2,(x-2)^2+y^2=k^2\right\}, & \text{if } k_{r-1}<k<k_{r-1}+\left[\sqrt{h_r}\right]\\ \{(0,0)\}, & \text{otherwise}\end{cases},$$

$$B_k=\begin{cases}\left\{(x,y)\in\mathbb{R}^2,(x+2)^2+y^2=k^2\right\}, & \text{if } k_{r-1}<k<k_{r-1}+\left[\sqrt{h_r}\right]\\ \{(0,0)\}, & \text{otherwise}\end{cases}.$$

Then $A_k \overset{WS_{\eta_2}^{\mu_1}(\theta,\mathcal{I})}{\sim} B_k$ for $\eta_2=\mu_1=\frac{1}{2}$ but $A_k \overset{WS_{\eta_1}^{\mu_2}(\theta,\mathcal{I})}{\nsim} B_k$, for $\eta_1=\frac{1}{4}$, $\mu_2=1$ and $L=1$. □

Corollary 2.8. *Let (X,ρ) be a metric space, $\theta=\{k_r\}$ be a lacunary sequence, $0<\eta_1\leq\eta_2\leq\mu_1\leq\mu_2\leq 1$ and A_k, B_k be non-empty closed subsets of X.*
(i) If $A_k \overset{WS_{\eta_1}(\theta,\mathcal{I})}{\sim} B_k$ implies $A_k \overset{WS_{\eta_2}(\theta,\mathcal{I})}{\sim} B_k$ for $\mu_1=\mu_2=1$.
(ii) If $A_k \overset{WS_{\eta_1}(\theta,\mathcal{I})}{\sim} B_k$ implies $A_k \overset{WS(\theta,\mathcal{I})}{\sim} B_k$ for $\eta_2=\mu_1=\mu_2=1$.

In [11], It is defined that the lacunary sequence $\theta^{'}=\{s_r\}$ is called a lacunary refinement of the lacunary sequence $\theta=\{k_r\}$ if $\{k_r\}\subseteq\{s_r\}$. In [13], the inclusion relationship between S_θ and $S_{\theta'}$ is studied.

Theorem 2.9. *Suppose $\theta'=\{s_r\}$ is a lacunary refinement of the lacunary sequence $\theta=\{k_r\}$. Let $I_r=(k_{r-1},k_r]$ and $J_r=(s_{r-1},s_r]$, $r=1,2,3,....$ If there exists $\epsilon>0$ such that for $0<\eta_1\leq\eta_2\leq\mu_1\leq\mu_2\leq 1$ and*

$$\frac{|J_j|^{\eta_2}}{|I_i|^{\eta_1}}\geq\epsilon \text{ for every } J_j\subseteq I_i\ ,$$

then $A_k \overset{WS_{\eta_1}^{\mu_2}(\theta,\mathcal{I})}{\sim} B_k$ implies $A_k \overset{WS_{\eta_2}^{\mu_1}(\theta',\mathcal{I})}{\sim} B_k$.

Proof. For any $\varepsilon > 0$ and every J_j, we can find I_i such that $J_j \subseteq I_i$; then we can write

$$\begin{aligned}\frac{1}{|J_j|^{\eta_2}}\left|\left\{k \in J_j : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|^{\mu_1} &= \left(\frac{|I_i|^{\eta_1}}{|J_j|^{\eta_2}}\right)\left(\frac{1}{|I_i|^{\eta_1}}\right)\left|\left\{k \in J_j : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|^{\mu_1} \\ &\leq \left(\frac{|I_i|^{\eta_1}}{|J_j|^{\eta_2}}\right)\left(\frac{1}{|I_i|^{\eta_1}}\right)\left|\left\{k \in I_i : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|^{\mu_2} \\ &\leq \left(\frac{1}{\epsilon}\right)\left(\frac{1}{|I_i|^{\eta_1}}\right)\left|\left\{k \in I_i : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|^{\mu_2},\end{aligned}$$

and so

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{|J_j|^{\eta_2}}\left|\left\{k \in J_j : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|^{\mu_1} \geq \delta\right\} \\ \subseteq\ &\left\{r \in \mathbb{N} : \left(\frac{1}{|I_i|^{\eta_1}}\right)\left|\left\{k \in I_i : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|^{\mu_2} \geq \delta\epsilon\right\} \in \mathcal{I}.\end{aligned}$$

This completes the proof. □

Theorem 2.10. *Suppose $\theta = \{k_r\}$ and $\theta' = \{s_r\}$ are two lacunary sequences. Let $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $r = 1, 2, 3, ...,$ and $I_{ij} = I_i \cap J_j$, $i, j = 1, 2, 3, ...$ If there exists $\epsilon > 0$ such that for $0 < \eta_1 \leq \eta_2 \leq \mu_1 \leq \mu_2 \leq 1$ and*

$$\frac{|I_{ij}|^{\eta_2}}{|I_i|^{\eta_1}} \geq \epsilon \text{ for every } i, j = 1, 2, 3, ..., \text{provided } I_{ij} \neq \varnothing,$$

then $A_k \overset{WS^{\mu_2}_{\eta_1}(\theta,\mathcal{I})}{\sim} B_k$ implies $A_k \overset{WS^{\mu_1}_{\eta_2}(\theta',\mathcal{I})}{\sim} B_k$.

Proof. Omitted. □

Theorem 2.11. *Let $\theta = \{k_r\}$ and $\theta' = \{s_r\}$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let η_1, η_2, μ_1 and μ_2 be such that $0 < \eta_1 \leq \eta_2 \leq \mu_1 \leq \mu_2 \leq 1$,*

(i) If

$$\lim_{r\to\infty} \inf \frac{h_r^{\eta_1}}{\ell_r^{\eta_2}} > 0 \tag{2.1}$$

then $A_k \overset{WS^{\mu_2}_{\eta_2}(\theta',\mathcal{I})}{\sim} B_k$ implies $A_k \overset{WS^{\mu_1}_{\eta_1}(\theta,\mathcal{I})}{\sim} B_k$,

(ii) If

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^{\eta_2}} = 1 \tag{2.2}$$

then $A_k \overset{WS^{\mu_2}_{\eta_1}(\theta,\mathcal{I})}{\sim} B_k$ implies $A_k \overset{WS^{\mu_1}_{\eta_2}(\theta',\mathcal{I})}{\sim} B_k$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$, $\ell_r = s_r - s_{r-1}$.

Proof. (*i*) Omitted.

(ii) Let $A_k \overset{WS_{\eta_1}^{\mu_2}(\theta,\mathcal{I})}{\sim} B_k$ and (2.2) be satisfied. Since $I_r \subset J_r$, for $\varepsilon>0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^{\eta_2}}\left|\left\{k\in J_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1} &= \frac{1}{\ell_r^{\eta_2}}\left|\left\{s_{r-1}<k\leq k_{r-1}:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\\
&\quad+\frac{1}{\ell_r^{\eta_2}}\left|\left\{k_r<k\leq s_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\\
&\quad+\frac{1}{\ell_r^{\eta_2}}\left|\left\{k_{r-1}<k\leq k_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\\
&\leq \frac{(k_{r-1}-s_{r-1})^{\mu_1}}{\ell_r^{\eta_2}}+\frac{(s_r-k_r)^{\mu_1}}{\ell_r^{\eta_2}}\\
&\quad+\frac{1}{\ell_r^{\eta_2}}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\\
&\leq \frac{k_{r-1}-s_{r-1}}{\ell_r^{\eta_2}}+\frac{s_r-k_r}{\ell_r^{\eta_2}}\\
&\quad+\frac{1}{\ell_r^{\eta_2}}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\\
&= \frac{\ell_r-h_r}{\ell_r^{\eta_2}}+\frac{1}{\ell_r^{\eta_2}}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\\
&\leq \frac{\ell_r-h_r^{\eta_2}}{h_r^{\eta_2}}+\frac{1}{h_r^{\eta_2}}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_2}\\
&\leq \left(\frac{\ell_r}{h_r^{\eta_2}}-1\right)+\frac{1}{h_r^{\eta_1}}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_2}
\end{aligned}$$

and so

$$\begin{aligned}
&\left\{r\in\mathbb{N}:\frac{1}{\ell_r^{\eta_2}}\left|\left\{k\in J_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\geq\delta\right\}\\
\subseteq\ &\left\{r\in\mathbb{N}:\frac{1}{h_r^{\eta_1}}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_2}\geq\delta\right\}\in\mathcal{I}
\end{aligned}$$

for all $r\in\mathbb{N}$. This implies that $A_k \overset{WS_{\eta_2}^{\mu_1}(\theta',\mathcal{I})}{\sim} B_k$. □

Theorem 2.12. *Let $\theta=\{k_r\}$ and $\theta'=\{s_r\}$ be two lacunary sequences such that $I_r\subseteq J_r$ for all $r\in\mathbb{N}$, η_1,η_2,μ_1 and μ_2 be fixed real numbers such that $0<\eta_1\leq\eta_2\leq\mu_1\leq\mu_2\leq1$. Let (2.1) holds, if $A_k \overset{WN_{\eta_2}^{\mu_2}[\theta',\mathcal{I}]}{\sim} B_k$ then $A_k \overset{WS_{\eta_1}^{\mu_1}(\theta,\mathcal{I})}{\sim} B_k$.*

Proof. For $0<\eta_1\leq\eta_2\leq\mu_1\leq\mu_2\leq1$ and $\varepsilon>0$, we have

$$\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|^{\mu_2}\geq\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\varepsilon^{\mu_1}$$

and

$$\begin{aligned}
\frac{1}{\ell_r^{\eta_2}}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|^{\mu_2} &\geq \frac{1}{\ell_r^{\eta_2}}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\varepsilon^{\mu_1}\\
&= \frac{h_r^{\eta_1}}{\ell_r^{\eta_2}}\frac{1}{h_r^{\eta_1}}\left|\left\{k\in I_r:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\varepsilon^{\mu_1}
\end{aligned}$$

and so

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^{\eta_1}}\left|\left\{k\in I_r:\left|\frac{d\left(x,A_k\right)}{d\left(x,B_k\right)}-L\right|\geq\varepsilon\right\}\right|^{\mu_1}\geq\delta\right\}$$
$$\subseteq\quad\left\{r\in\mathbb{N}:\frac{1}{\ell_r^{\eta_2}}\sum_{k\in J_r}\left|\frac{d\left(x,A_k\right)}{d\left(x,B_k\right)}-L\right|^{\mu_2}\geq\varepsilon^{\mu_1}\delta\frac{h_r^{\eta_1}}{\ell_r^{\eta_2}}\right\}\in\mathcal{I}.$$

Since (2.1) holds it follows that if $A_k \overset{WN_{\eta_2}^{\mu_2}[\theta',\mathcal{I}]}{\sim} B_k$, then $A_k \overset{WS_{\eta_1}^{\mu_1}(\theta,\mathcal{I})}{\sim} B_k$. □

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